Course 3 - Beijing - Nov. 2019

Zeros of holomorphic functions of the variable p

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Recall: $E \nearrow [E:\mathbb{Q}_p] < +\infty$, $E \searrow \mathbb{F}_q((\pi))$ $\mathbb{F}_q = \mathcal{O}_E/\pi$

$F|\mathbb{F}_q$ perfectoid

$$A = \begin{cases} W_{\mathcal{O}_E}(\mathcal{O}_F) \\ \mathcal{O}_F[[\pi]] \end{cases}$$

$$Y = \mathrm{Spa}(A, A) \setminus V(\pi[\varpi])$$

$$B := \mathcal{O}(Y) = \text{Completion of } A\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] \text{ w.r.t. } (|\cdot|_\rho)_{\rho \in]0,1[}$$

E-Fréchet algebra

$$f = \sum_n [x_n]\pi^n \in A\left[\tfrac{1}{\pi}, \tfrac{1}{[\varpi]}\right]$$

$$|f|_\rho = \sup_n |x_n|\rho^n$$

$$= q^{-v_r(f)} \quad \text{if } \rho = q^{-r}$$

$$v_r(f) = \inf_{n \in \mathbb{Z}} v(x_n) + nr$$

For $f \in B$ one can define

$\mathrm{Newt}(f) =$ decreasing convex polygon breakpoints

x-coord. $\in \mathbb{Z}$

$=$ inverse Legendre transform of the concave polygon

$]0,+\infty[\ni r \longmapsto v_r(f)$

If $f = \sum_{n \gg -\infty} [x_n] \pi^n \in A\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$

$\mathrm{Newt}(f) =$ decreasing convex hull of $(n, v(x_n))_{n \in \mathbb{Z}}$

$\mathrm{Newt}(fg) = \mathrm{Newt}(f) \underbrace{\circledast}_{\text{concatenation}} \mathrm{Newt}(g)$

The case $E = \mathbb{F}_q((\pi))$

$$Y = \mathbb{D}_F^* = \{0 < |\pi| < 1\} \subset \mathbb{A}_F^{1,\mathrm{ad}}$$

$$B = \mathcal{O}(Y)$$

classical Tate points

$$|Y|^{cl} = \{ z \in \bar{F} \,/\, 0 < |z| < 1 \} \,/\, Gal(\bar{F}|F)$$

$$= \{ P \in F[T] \,/\, P \text{ irreducible and its roots } z \text{ satisfy } 0<|z|<1 \} \,/\, F^\times$$

$$= \{ P \in \mathcal{O}_F[T] \,/\, P \text{ unitary irreducible}, \ 0 < |P(0)| < 1 \}$$

Def: $f = \sum_{n \geq 0} x_n T^n \in \mathbb{A}$ is distinguished / primitive of deg. $d > 0$ if $x_0 \neq 0$, $x_0, \dots, x_{d-1} \in \mathfrak{m}_F$ and $x_d \in \mathcal{O}_F^\times$

Weierstrass factorization: f primitive deg. d

$$f = u \cdot P \quad \text{unique}$$

$u \in \mathbb{A}^\times = \mathcal{O}_F[[T]]^\times$

P: unitary deg. $d \in \mathcal{O}_F[T]$, $0 < |P(0)| < 1$

$$\Rightarrow |Y|^{cl} = \{\text{primitive irred.}\} / \mathbb{A}^\times$$

The case $E|\mathbb{Q}_p$: Same def.

Def: $f = \sum_{n\geq 0} [x_n]\pi^n \in A = W_{O_E}(O_F)$ is primitive if $x_0 \neq 0$, $x_0, \ldots, x_{d-1} \in m_F$, $x_d \in O_F^\times$

Rem: f primitive deg. d $\iff \begin{cases} f \bmod \pi \neq 0 \in O_F \\ f \bmod W_{O_E}(m_F) \neq 0 \in W_{O_E}(k_F) \end{cases}$

$$\deg(f) = v_\pi\left(f \bmod W_{O_E}(m_F)\right)$$

$\hookrightarrow \deg(fg) = \deg(f) + \deg(g)$

$$\boxed{\textbf{Def:}\ |Y|^{cl} = \mathrm{Prim}/A^\times}$$

③

Intermezzo: Perfectoid fields

Def. K complete $|\cdot|: K \to \mathbb{R}_+$ is perfectoid if $\exists\, \varpi \in K$ s.t. $0 < |\varpi| < 1$ and

(1) $\varpi^p \mid p$

(2) $\mathrm{Frob}: \mathcal{O}_K/\varpi \longrightarrow \mathcal{O}_K/\varpi$ is surjective

Ex: $\mathrm{Char}(K) = p$. Then K perfectoid $\Leftrightarrow$ K perfect

Tilting: K perfectoid.

$$K^\flat = \left\{ \left(x^{(n)}\right)_{n \geq 0} \in K^{\mathbb{N}} \;\middle|\; \left(x^{(n+1)}\right)^p = x^{(n)} \right\}$$

$\overset{\shortparallel}{x}$ set $x^\# := x^{(0)}$

$$(xy)^{(n)} = x^{(n)} y^{(n)}$$

$$(x+y)^{(n)} = \lim_{h \to +\infty} \left(x^{(n+h)} + y^{(n+h)}\right)^{p^h}$$

Then $(K^\flat, |\cdot|)$ is a char. p perfectoid field.

$$|x| := |x^\#|$$

Moreover $\mathcal{O}_{K^\flat} \xrightarrow{\sim} \varprojlim_{\text{Frob}} \mathcal{O}_K/p$ iso. of rings

$$(x^{(n)})_{n\geq 0} \longmapsto (x^{(n)} \bmod p)_{n\geq 0}$$

$$\left(\lim_{k\to+\infty} (\hat{y}_{n+k})^{p^k}\right)_{n\geq 0} \longleftarrow y_n$$

$\hat{y}_{n+k}$ ↳ any lift of y_n in $\mathcal{O}_K$

Exm $K|\mathbb{F}_p \Rightarrow K^\flat = K$.

<u>Intermezzo</u>: Fontaine's θ morphism

$R = \underbrace{p\text{-adic ring}}$, $R^\flat = \varprojlim_{\text{Frob}} R/pR$

↳ p-adically separated complete

$$R^\flat = \{(x^{(n)})_{n\geq 0} \in R^{\mathbb{N}} \,/\, (x^{(n+1)})^p = x^{(n)}\}$$

w.r.t. $(xy)^{(n)} = x^{(n)} y^{(n)}$

$$(x+y)^{(n)} = \lim_{k\to\infty} \left(x^{(n+k)} + y^{(n+k)}\right)^{p^k}$$

Prop. The two functors

$$\{p\text{-adic rings}\} \underset{W(-)}{\overset{(-)^\flat}{\rightleftarrows}} \{\text{perfect } \mathbb{F}_p\text{-algebras}\}$$

are adjoint where the adjunction morphisms are given by

$$R \xrightarrow{\sim} W(R)^\flat$$

$$x \longmapsto \left([x^{1/p^n}]\right)_{n\geq 0}$$

and

$$\begin{cases} \theta : W(R^\flat) \longrightarrow R \\ \sum_{n\geq 0} [x_n] p^n \longmapsto \sum_{n\geq 0} x_n^\# p^n \end{cases}$$

Rem: R p-adic s.t. $\mathrm{Frob}_{R/pR}$ surjective.

$\Rightarrow \theta : W(R^\flat) \longrightarrow R$ is surjective mod p
and thus surjective
by Nakayama

$\Rightarrow R =$ quotient of $W(R^\flat)$.

Back to primitive elements

Th: $\xi \in \mathbb{A}$ primitive degree d. irreducible.
$\mathcal{O}_K = \mathbb{A}/\xi$ and $K = \mathcal{O}_K\left[\frac{1}{\varpi}\right]$

$\theta : \mathbb{A} \twoheadrightarrow \mathbb{A}/\xi$

(1) $K|E$ is a perfectoid field w.r.t. ring of integers $\mathcal{O}_K$

$|\theta([x])| = |x|$

(2) The morphism $\mathcal{O}_F \to \mathcal{O}_K^\flat$

$x \longmapsto \left(\theta\left(\left[x^{1/p^n}\right]\right)\right)_{n\geq 0}$

induces an extension $K^\flat | F$ s.t.

$$\boxed{[K^\flat : F] = d}$$

(3) For $d=1$ this induces

$$\underset{\underset{\mathrm{Spec}(\mathbb{A})}{\cap}}{\mathrm{Prim}^{\deg=1}} / \mathbb{A}^\times \xrightarrow{\sim} \underbrace{\left\{ K | E \text{ perfectoid with } F \xrightarrow{\sim} K^\flat \right\} / \sim}_{\text{untilts of } F}$$

$$(\xi) \longmapsto \mathbb{A}[\tfrac{1}{\pi}] / \xi$$

$$\ker \theta \longleftarrow K$$

~~This~~

<u>Def.</u> $|Y|^{cl} \subset |Y|$
$$\parallel$$
$\{ V(\xi) \mid \xi \in \mathbb{A} \text{ irreducible primitive} \}$

For $y \in |Y|^{cl}$, $k(y)|E$ perfectoid, $[k(y)^\flat : F] < +\infty$.

Th. F alg. closed $\Rightarrow$

(1) $\forall \xi$ irred. primitive $A[\frac{1}{\pi}]/\xi$ is alg. closed

(2) " " $\deg(\xi) = 1$

(1) Purity theorem: K alg. closed $\Leftrightarrow K^\flat$ alg. closed ($\overset{\text{easy}}{\Rightarrow}$)

(2) Weierstrass factorization: $\forall \xi$ primitive deg. d

$$\xi = \underset{\underset{A^\times}{\cap}}{u} \cdot (\pi - [a_1]) \cdots (\pi - [a_d]), \quad a_i \in \mathfrak{m}_F \setminus \{0\}$$

not unique contrary to $E = \mathbb{F}_q((\pi))$

Def: $y \in |Y|^{cl}$ $\quad y = V(\xi)$

$$\xi = \sum_{n \geq 0} [x_n] \pi^n \quad \text{deg. } d$$

Set $|y| = |x_0|^{1/d} \in]0,1[$

$$= |\pi(y)|$$

$I \subset]0,1[$ interval $|Y_I|^{cl} = \{ y \in |Y|^{cl} \mid |y| \in I \}$

~~Thm (Localization of zeros via Newton polygons):~~

* $y \in |Y|^{cl}$ $\quad y = V(\xi)$

$B^+_{dR,y}$ = ξ-adic completion of $A[\frac{1}{p}, \frac{1}{[\varpi]}]$

$$= \widehat{\mathcal{O}}_{Y,y}$$

= D.V.R. with uniformizing element ξ and residue field $k(y)$.

$\mathrm{ord}_y : B^+_{dR,y} \to \mathbb{Z} \cup \{+\infty\}$ valuation

<u>Th</u> (Localization of zeros): $f \in B \setminus \{0\}$

$\{ |y| \mid y \in |Y|^{cl} \text{ and } f(y) = 0 \}$ with multiplicity $\mathrm{ord}_y(f)$

$\|$

$\{ \text{slopes of } \mathrm{Newt}(f) \}$

Idem with B_I and Newt_I

<u>Th</u>: I compact $\Rightarrow$ B_I is a P.I.D. wt. $\mathrm{Spm}(B_I) = |Y_I|^{cl}$

The divisor of an holomorphic function

$I \subset]0,1[$ intervalle

$$\mathrm{Div}^+(Y_I) = \begin{cases} \mathbb{N}[|Y_I|^{cl}] & \text{if } I \text{ compact} \\ \varprojlim_{J \subset I \text{ compact}} \mathrm{Div}^+(Y_J) \end{cases}$$

locally finite effective divisors on Y_I

on each compact open subset

$$= \left\{ D = \sum_{y \in |Y_I|^{cl}} m_y [y] \;\middle|\; \forall U \subset Y_I \text{ q.c. } \; U \cap \mathrm{supp}(D) \text{ finite} \right\}$$

$f \in B_{I, \{0\}}$ $\quad \mathrm{div}(f) = \sum_{y \in |Y_I|^{cl}} \mathrm{ord}_y(f)\,[y] \in \mathrm{Div}^+(Y_I)$

Prop: $f \in B_I^\times \iff \mathrm{div}(f) = 0$ i.e. f has no zero on $|Y_I|^{cl}$

$$\Rightarrow \boxed{B_I \setminus \{0\} / B_I^\times \xhookrightarrow{\mathrm{div}} \mathrm{Div}^+(Y_I)}$$

Of course: I compact $\Rightarrow B_I \setminus \{0\} / B_I^\times \xrightarrow{\sim} \mathrm{Div}^+(Y_I)$

since B_I is a P.I.D.

Rem: It is false that this is a bijection for $I =]0,1[$.

In the case $E = \mathbb{F}_q((\pi))$ this is true iff F is spherically complete (Lazard)

Prop: $\rho \in]0,1[$ then any effective divisor on $Y_{]0,\rho]}$ is principal i.e. $B_{]0,\rho]} / B_{]0,\rho]}^\times \xrightarrow{\sim} \mathrm{Div}^+(Y_{]0,\rho]})$

$\rightarrow$ Weierstrass product: $D = \sum_{n \geq 0} [y_n] \in \mathrm{Div}^+(Y_{]0,\rho]})$

$\lim_{n \to +\infty} |y_n| = 0$.

$y_n = V(\xi_n)$

$\uparrow$ irred. primitive

Can suppose $\xi_n \bmod W(k_F) = \pi^{\deg(\xi_n)}$

Then $f = \prod_{n\geq 0} \dfrac{\xi_n}{\pi^{\deg(\xi_n)}} \in B_{]0,\rho]}$ C.V.

$\mathrm{div}(f) = D$.

Ex: Fatg-closed $\xi_n = \pi - [a_n]$ $\quad a_n \underset{n\to+\infty}{\longrightarrow} 0$

$\prod_{n\geq 0}\left(1 - \dfrac{[a_n]}{\pi}\right)$ C.V.

~~The following~~

Corollary: * $B_{]0,\rho]}$ is a Bezout ring for $\rho \in]0,1[$.

$* \boxed{\mathcal{R} = \varinjlim_{\rho \to 0,\, \rho > 0} B_{]0,\rho]} \text{ is Bezout}}$

$\hookrightarrow$ germs of holo. fct. around $\pi = 0$